The Anchor Point Lemma

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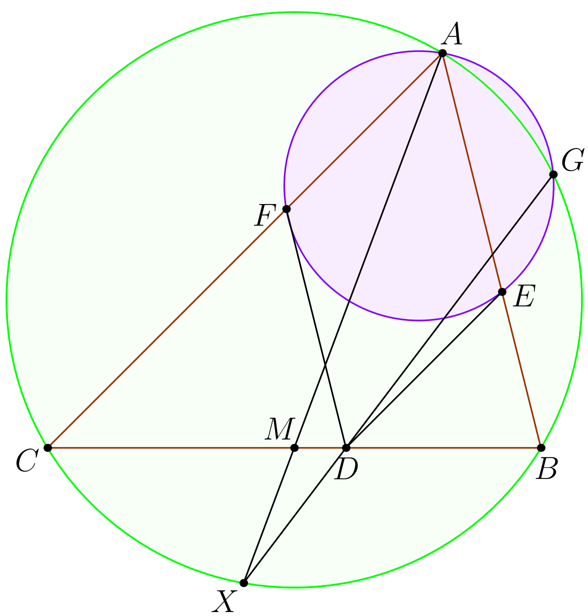
§1 The Config

Problem 1.1 (AoPS)

Given a triangle $\triangle ABC$, let D be an arbitrary point on BC, then let DE and DF be parallel to AC and AB respectively. Let (AEF) intersect (ABC) at G, let GD intersect (ABC) at X. Prove, that if M is the midpoint of AC, then A, X and M are colinear.



This theorem can be proven in various ways, the first way and arguably most beautiful way is using the Butterfly theorem, (thanks @KrazyNumberMan)

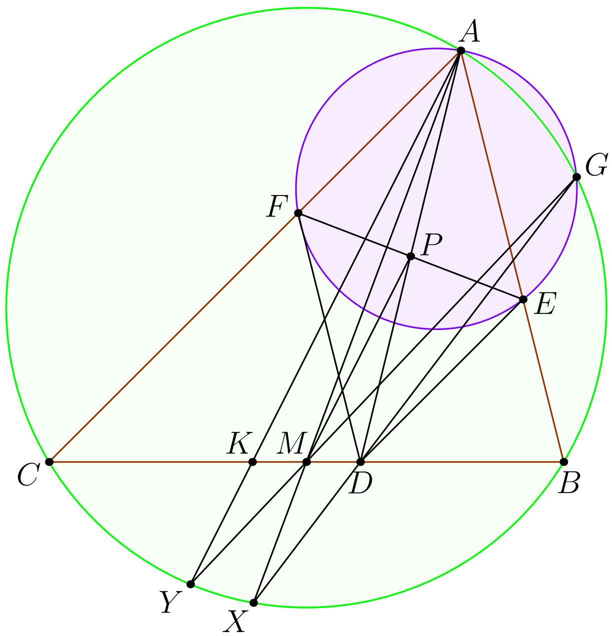
Proof. Let Y be the intersection of MG with (ABC) and let K be the intersection of AY with BC and $P = EF \cap AD$. Then,

Lemma 1.1 $MP \parallel AK$

indeed, since G is the Miquel point of $CFEB$ it must be that $\triangle GEB \sim \triangle GPD \sim \triangle GFC$, consequently,

$$\measuredangle GMP = \measuredangle GBA = \measuredangle GYA$$

thus implying the desired result. □



Since P is the midpoint of AD and $MP \parallel AK$ it must be that PM is the midline in $\triangle AKD$, thus M is the midpoint of KD. Thus, by the Butterfly theorem it must be that X, D, G are colinear. ■

Another proof of the original theorem involves constructing the symmedian, the proof in itself is not particularly interesting, however the results shown are somewhat reasonable.

Proof. Let us prove that,

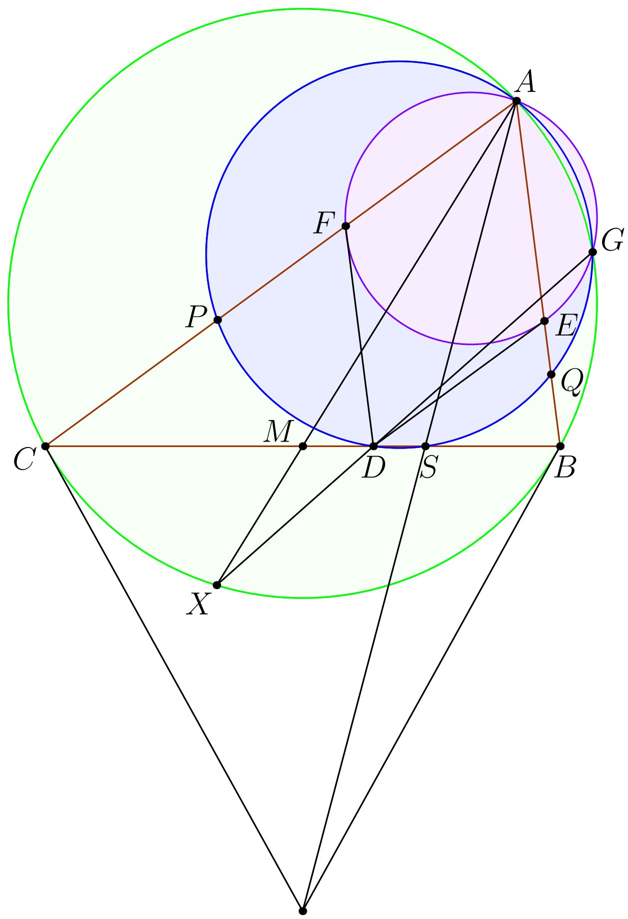
Lemma 1.2 $S \in (AGD)$, where S is the foot of the symmedian from A onto BC.

Using coaxility lemma, all we have to prove is that,

$$\frac{PE}{BP} = \frac{FQ}{QC}$$

Then this would mean that (APQ) passes through G, where we define P and Q to be the intersection points of (ADS) with AB and AC. Notice,

$$BE = AB - EA = AB - DF = AB - AB \cdot \frac{CD}{BC} = AB \cdot \frac{BD}{BC}$$



Now let us try calculating the value of BP, this can be done through the Power of the Point,

$$BP = \frac{BS \cdot BD}{BA}$$

Thus,

$$\frac{PE}{BP} = \frac{BE - BP}{BP} = \frac{BE}{BP} - 1 = \frac{AB \cdot \frac{DB}{BC}}{\frac{BS \cdot BD}{BA}} - 1 = \frac{AB^2}{BS \cdot BC} - 1$$

We want to show this is the same as,

$$\frac{FQ}{QC} = \frac{AC^2}{CS \cdot BC} - 1$$

(same logic for this expression). Let us use the formula for the length of BS (because the symmedian is the isogonal conjugate of the median),

$$BS = \frac{AB^2}{AC^2} \cdot CS$$

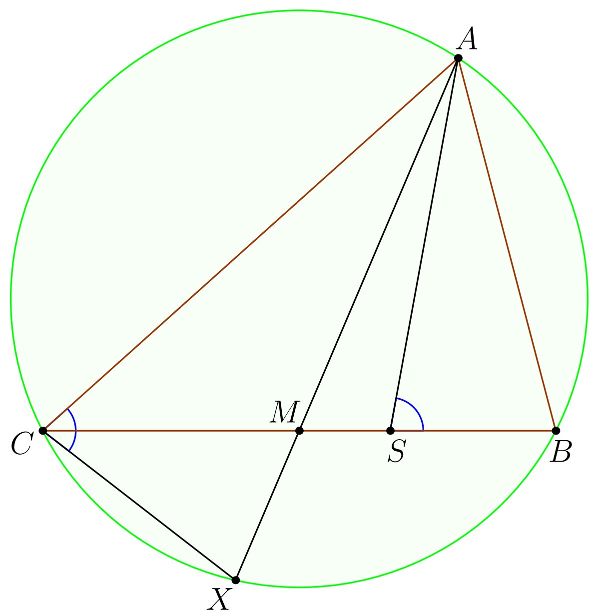
Thus,

$$\frac{PE}{BP} = \frac{AB^2}{BS \cdot BC} - 1 = \frac{AB^2}{\frac{AB^2}{AC^2} \cdot CS \cdot BC} - 1 = \frac{AC^2}{CS \cdot BC} - 1 = \frac{FQ}{QC}$$

This finishes the proof of the lemma, thus $ASDG$ is cyclic. *square*

It is quite well known that if X' is the intersection of AM with (ABC), then $\angle ABX' = \angle ASC$.

Lemma 1.3 $\angle ABX' = \angle ASC$



this is just obvious because $\triangle ABX' \sim \triangle ASC$, due to $\angle ABX' = \angle SAC$ and $\angle AX'B = \angle ACS$. □

Now going back to our problem, notice that,

$$\angle ASC = 180 - \angle AGX = \angle ABX$$

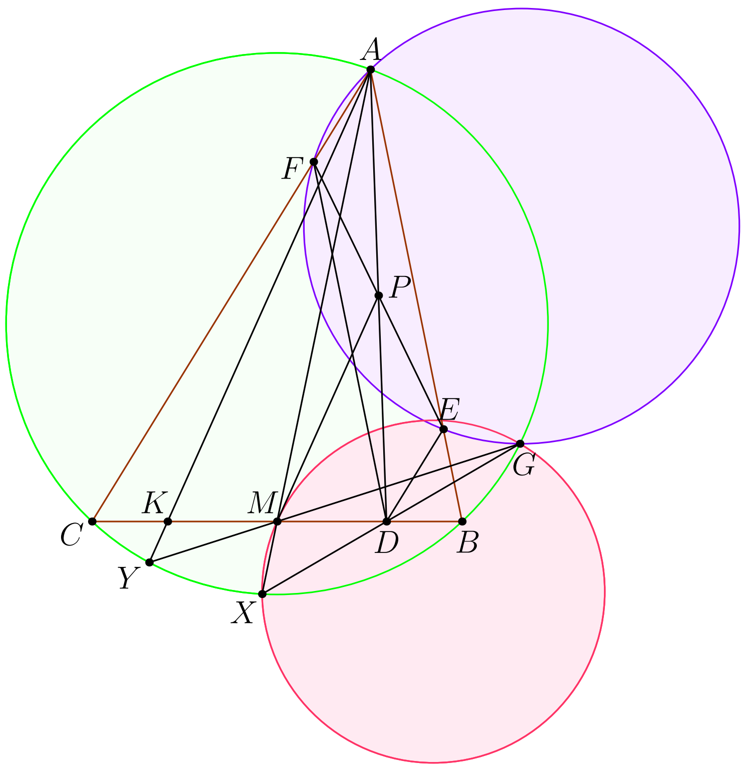
Because only one point satisfies such condition it must mean that $X' = X$ in other words, A, M and X are colinear.

■

§2 The Conflict

Now, given this powerful configuration let extend it and consider some other problems,

> **Problem 2.1** (AoPS)
>
> Prove that MP is tangent to (XMG).

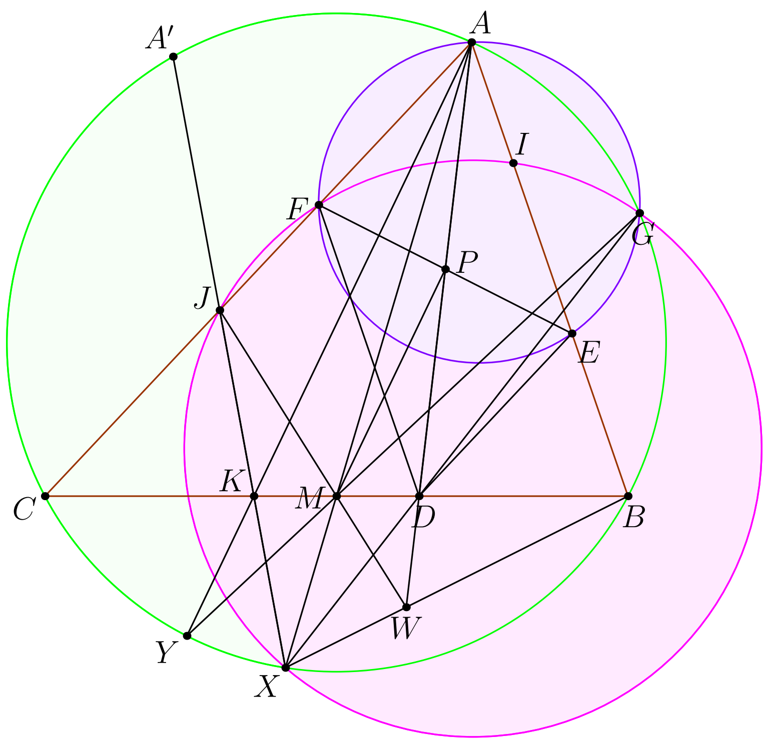


Proof. Notice,

$$\measuredangle GMP = \measuredangle GYA = \measuredangle GXA$$

which proves the desired tangency. ■

> **Problem 2.2** (AoPS)
>
> Let J be the intersection of (XIG) with AC. Prove that JM and AD intersect on XB.



Proof. (thanks @keglesnit)

Let $J = XK \cap AC$ and let $W = AD \cap XB$, then let us prove that JW passes through M. Let A' be the second intersection of XK with (ABC). Then,

$$(A', G; B, C) \stackrel{X}{=} (K, D; B, C) \stackrel{A}{=} (AK \cap (AEF), AD \cap (AEF); F, E)$$
$$\stackrel{P}{=} (G, A; E, F) = (A, G; F, E)$$

> **Note** The fact that $AK \cap (AEF), P$ and G lie on one line directly follows from angle chase similar to that done in the proof of the Anchor Point Lemma

This implies that $GBCA'$ is similar to $GEFA$. Thus,

$$\angle JXG = \angle A'XG = \angle AFG$$

consequently it must be that J is the intersection of (XEG) with AC.

Now, all that is left to show that JW passes through M.

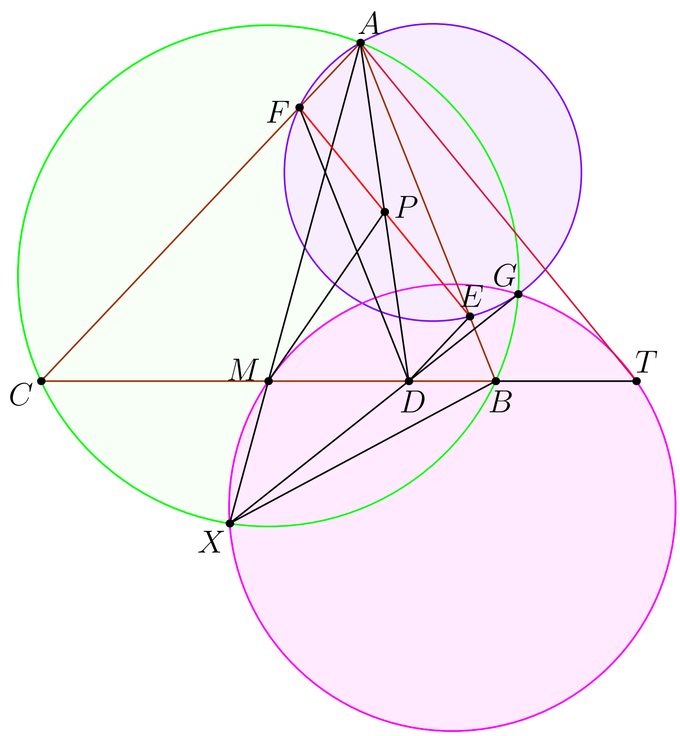
Let $J' = WM \cap AC$, then by Desargues Involution Theorem (DIT) applied to X, W, J' and A, there exists an involution swapping $B \leftrightarrow C, M \leftrightarrow M, D \leftrightarrow$

$XJ' \cap BC$. Consequently the involution must be a reflection over M, thus $XJ' \cap BC = K$, thus $J' = J$.

Consequently it must be that AD, XB and JM are concurrent. ■

Problem 2.3 (AoPS)

Let T be the intersection of (XMG) and BC. Prove that AJ and EF are parallel.



Proof. Notice, $(E, F; P, \infty_{EF}) = -1$, consequently projecting from A we obtain,

$$(B, C; D, U) = -1$$

where U is the intersection of a line parallel to EF through A with BC. However, since $XMGT$ is cyclic it must be that $(B, C; D, T) = -1$. (since $DT \cdot DM = DG \cdot DX = DB \cdot DC$) Consequently $T = U$, thus AT is parallel to EF. ■

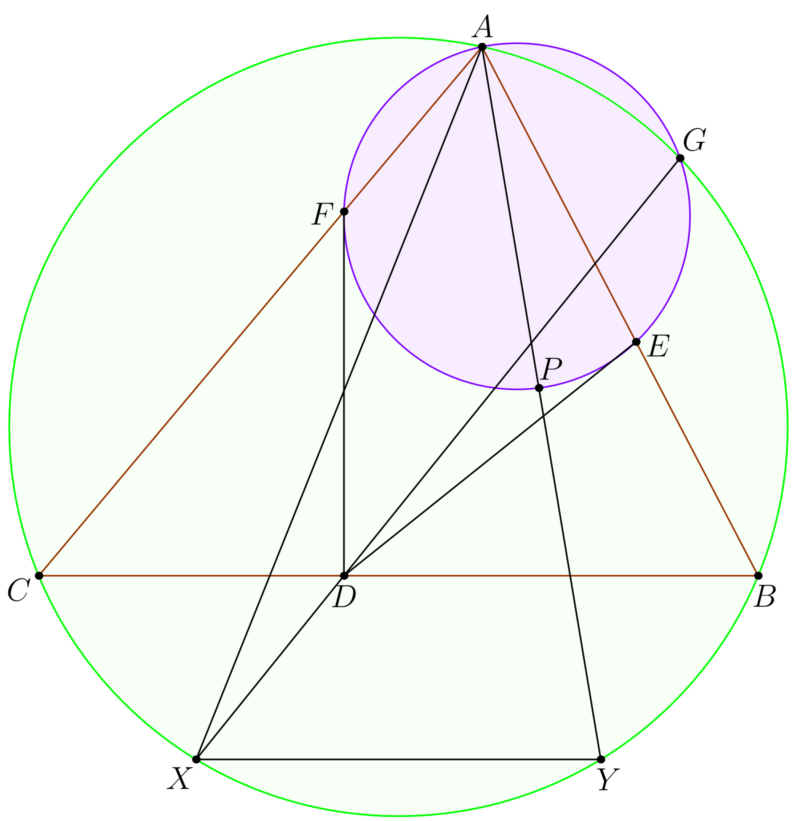
§3 The Generalization

While the original statement is useful, it can rarely be used in problem due to a rather peculiar condition on parallel lines. Thankfully there exists a far more useful generalized of the theorem,

Problem 3.1 (Generalized Anchor Point Lemma)

Let D be an arbitrary point on BC in $\triangle ABC$. Let there be two fixed directions l_1 and l_2. Let E and F be the intersection of two lines through D parallel to l_1 and l_2 with AB and AC. Let G be the intersection of (AFE) and (ABC). Prove that DG passes through a constant point on (ABC).

Also prove that the circles (AFE) are all coaxial and the second constant point is the intersection of the circle with the line isogonal to AX in $\angle CAB$ with (AFE).



Proof. Maybe it is possible to extend the synthetic approach described earlier, however it is far more simpler to use the Cool Ratio Lemma. Let $f(P) = \frac{BP}{CP}$, then,

$$f(X) = \frac{f(D)}{f(G)} = \frac{f(D)}{EB/CF} = \frac{CF \cdot BD}{EB \cdot DC} = \text{const}$$

thus since $f(X)$ is constant it must be that X is constant. ∎

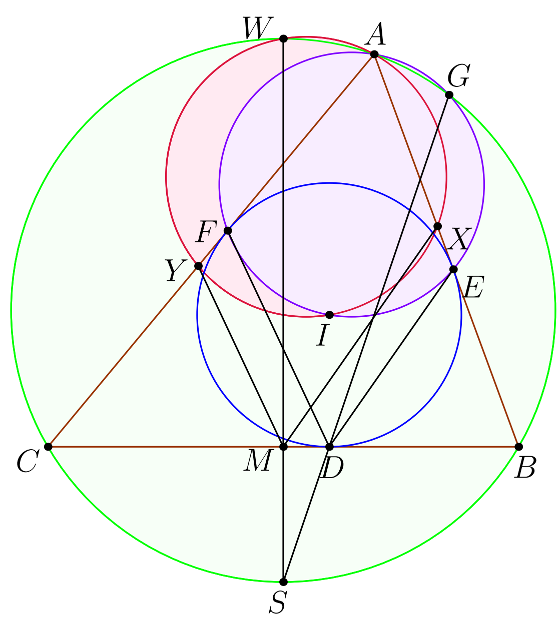
§4 The Anchor Point Method

Now I will outline a technique which is sometimes very powerful in simplifying a problem which involves some intersection of some circle with (ABC). Let us go through some well known configurations and attack them with the Generalized Anchor Point Lemma.

4.1 Sharky-Devil

Problem 4.1 (Sharky-Devil Configuration)

Let (I) be the incircle of $\triangle ABC$, let D, E, F be the tangency points of (I) with BC, AB and AC, respectively. Let G be the second intersection of (AFE) with (ABC). Let S be the midpoint of the arc BC. Prove that S, D and G are colinear.



Proof. Let us introduce M the midpoint of BC and W the midpoint of the larger arc BC. Then, let X and Y be the intersections of the lines through

M parallel to DE and DF with AB and AC, respectively. By the Generalized Anchor Point Lemma all that is left to prove is that $AWXY$ is cyclic.

Notice, since,

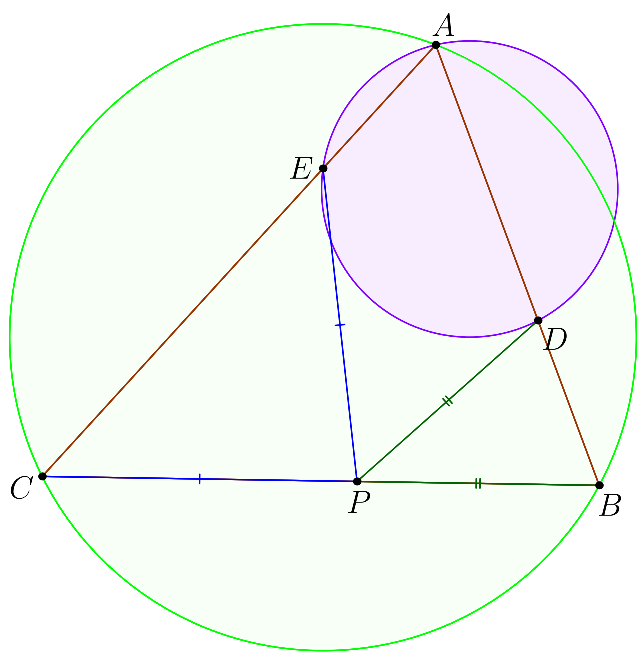
$$\measuredangle WXY = \measuredangle WAC$$
$$\measuredangle XYW = \measuredangle XAW$$

which implies that $\angle WAC = 180 - \angle BAW$, however this is only true for W being the midpoint of the larger arrc BC. Thus $WAXY$ is cyclic which proves one of the properties of the Sharky-Devil point. (Amusingly I lies on this circle as well due to $I \in (AFE)$ and I lying on the angle bisector of $\angle CAB$). ■

4.2 Problems

Problem 4.2 (USA TST 2012 P1)

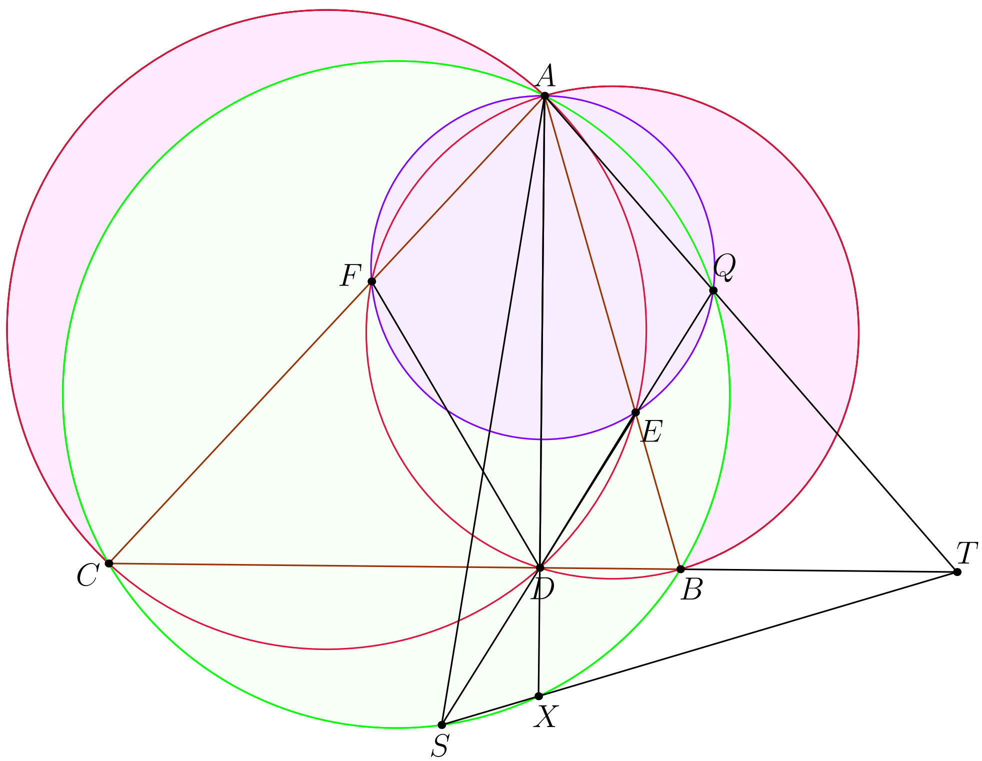
In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.



Proof. Since as we move P the lines EP and PD are parallel to two fixed directions, thus by the Generalized Anchor Point Lemma it must be that (AED) passes through a fixed point. ■

Problem 4.3 (ELMO 2013 Shortlist G3)

n $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.



Proof. Notice, FD and DE point in constant directions, since $\angle CDF = \angle A = \angle EDB$. Thus, by the Generalized Anchor Point Lemma all we need to do is show for one position of D that (AFE) passes through some fixed point on the median. Let us fix D to be the foot of the altitude from A to BC. Let S be the intersection of the symmedian from A with (ABC), then, (it is well known, however the proof is outlined below)

Lemma 4.1 S, D, Q are colinear.

indeed, since $ABSC$ is harmonic, by projecting from T it must be that $QBCX$ is harmonic, consequently projecting from D we obtain that Q goes to a point

W on (ABC) such that $ABCW$ is harmonic, thus $W = S$, thus Q, D, S are colinear. $\square$

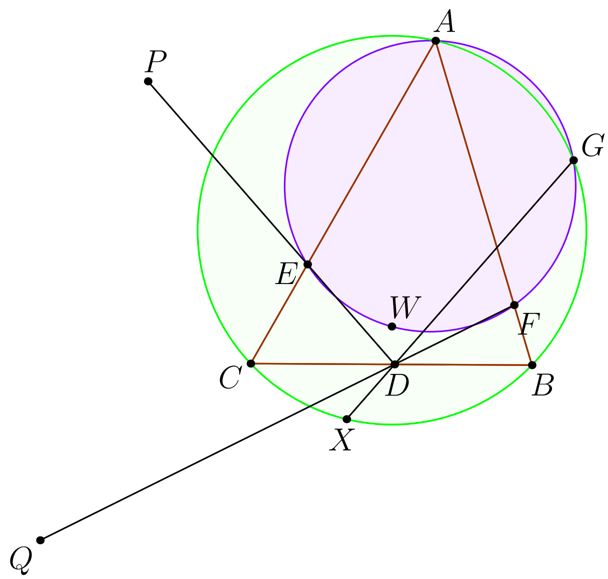
Now, by the Generalized Anchor Point Lemma since S, D, Q are colinear, it must be that (AEF) passes through a fixed point lying on the isogonal line to AS in $\angle CAB$ which is the median. $\blacksquare$

§5 Advanced Anchor Point Lemma

The problem with the Anchor Point Lemma and its generalization that it assumes that the lines stay parallel with respect to each other. The motivation for this generalization is that this can be interpreted as a pencil of lines through some point at infinity, so what happens if we move that point into $\mathbb{R}^2$? It turns out not any two pairs of points will keep the theorem true, however each point given a point on (ABC) has precicely one *conjugate* (which will be referenced as the *Anchor Point Conjugate* further on) which preserves this theorem.

Problem 5.1 (AoPS)

Let P be an arbitrary fixed point and X an abitrary fixed point on (ABC). Let D be an abritrary point on BC. Let PD intersect ACat E. Let XD intersect (ABC) at G. Let (AEG) intersect AB a second time at F. Prove that the line DF passes through a constant point Q as D moves on BC, and that (AEF) passes through a fixed point W.



The Generalized Anchor Point Lemma is simply a special case of the Advanced Anchor Point Lemma where P lies on the line at infinity, then it simply claims that the *Anchor Point Conjugate* lies on the line at infinity as well.

Interestingly,

> **Theorem 5.1** The *Anchor Point Conjugate* is projective, i.e. it preserves cross-ratios.

Proof. The map can be constructed like this, let D_1 be on BC such that $PD_1 \parallel AB$ and let k be the line through D_1 parallel to AC. Let D_2 be on BC so A, X, D_2 are colinear.

Then if $E = PX \cap AC$ and F is such that $EF \parallel BC$. Then, $Q = k \cap FD_2$.

Consequently it must be that if P moves projectively, Q moves projectively as well. ■

§6 Conclusion

I believe that the Generalized Anchor Point Method is quite powerful in problems invovling some type of intersection of (ABC) with a circle passing through A with well defined intersections with AB and AC.

> **Note** (TODO) Additional problems that can be solved using this method will be added to this document as I come across them.