Sawayama's Lemma and Thébault's circles

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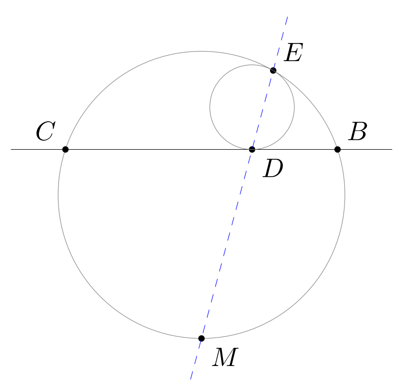
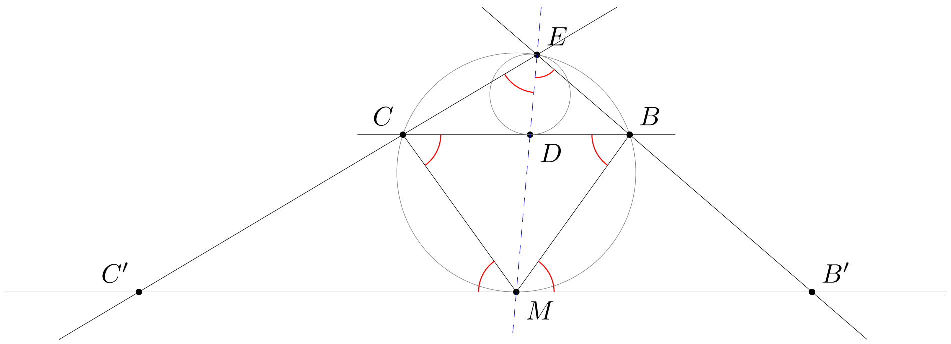
This paper compiles a collection of geometric proofs and related constructions that center around Sawayama's Lemma and Thébault's theorem. The document begins with the presentation and proof of the Shooting Lemma, which establishes a relationship between a chord in a circle, a tangent circle, and the midpoint of the larger arc. Using this foundational lemma, the proofs of Sawayama's Lemma and Verrier's Lemma follow, demonstrating the collinearity of specific points associated with inscribed triangles and tangent circles. The final section extends these results to prove Thébault's theorem, generalizing the principles of Sawayama's and Verrier's Lemmas to a broader context involving two tangent circles.

1 The Shooting Lemma

Theorem 1. *Consider the chord BC in the circle Ω. Let the circle ω touch BC at a point D and the circle Ω at a point E. Prove that the line DE passes through M, the middle of the larger arc $\overset{\frown}{BC}$.*



Proof. The proof is quite trivial, simply consider the homothety centered at E, which transforms ω into Ω. Then, B is mapped to B' and C to C', where B' and C' are the intersections of CE and BE with the tangent from M to Ω respectevely.



Then, $\angle BCM = \angle CMC'$ due to $CB \parallel C'B'$, and $\angle CBM = \angle CMC'$ because $C'B'$ touches Ω and finally $\angle CBM = \angle BMB'$. Thus, $\angle CMC' = \angle BMB'$, however due to Ω touching $C'B'$ we know that $\angle CEM = \angle CMC'$ and $\angle BEM = \angle BMB'$, consequently $\angle CEM = \angle MEB$. In other words ME is the bisector of $\angle CEB$ which means that M is the middle of the larger arc $\overset{\frown}{BC}$.

$\blacksquare$

In fact, due to $\angle BEM = \angle DBM$ we conclude that (EDB) touches BM, consequently,

$$pow_{(EDB)}M = MD \cdot ME = MB^2 \tag{1}$$

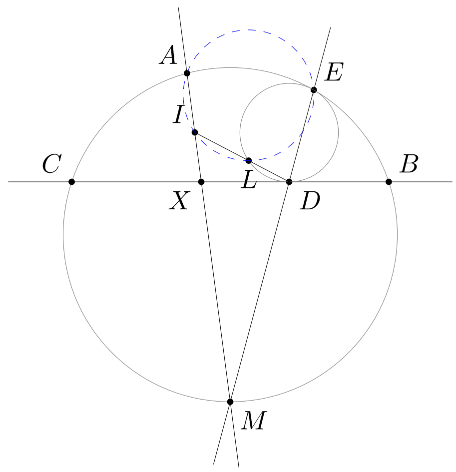
And due to M being the middle of $\overset{\frown}{BC}$ it must be that $CM = MB$, thus, $MB^2 = MC \cdot MB$. Combining these results we get a nice formula,

$$MC \cdot MB = MD \cdot ME \tag{2}$$

The figure can also show a lot of interesting and fundemental properties if one performs an inversion centered at the point M with a radius of $MB = MC$. Then through this process $\Omega \leftrightarrow BC$ and because ω must continue to touch $inv(BC)$ and $inv(\Omega)$ (in other words Ω and BC) and it must be in the same angle from M, it must be the case that $\omega \to \omega$ under the inversion. Consequently, it must mean that $D \leftrightarrow E$ and thus M, D and E are colinear. Another consequence of such argument is that the length of the tangents from M to ω are equal to $MB = MC$.

Now let us consider the following, a bit stronger statement,

Theorem 2. *Let A be an arbitrary point on the arc $\overset{\frown}{CEB}$ and let I be an arbitrary point on AM. Let L be the intersection of ID and ω, prove that $AILE$ is cyclic.*



Proof. This is also quite a trivial statement, noticing from the previous statement that $CM^2 = MX \cdot AM$ and $CM^2 = MD \cdot ME$ we can conclude that $MX \cdot MA = MD \cdot ME$ which by the power of the point M concludes that $AXDE$ is cyclic. Now, all that is left to notice is that, $\angle ELD = \angle EDB$ due to DB touching ω and $\angle EDB = \angle EAX$ due to $AXED$ being cyclic, thus $\angle IAE = \angle ELD$ and $AILE$ is cyclic.

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It is a bit interesting to see the behaviour of $(AILE)$ as one moves I along AM. When $I = X$ we get an already proven statement that $AXDE$ is cyclic and when $I = A$ we see that $(AILE)$ touches ED. However, there is a more important position of I which has the following property.

Lemma 1. *If $MI = BM = CM$, then AL tangent to ω.*

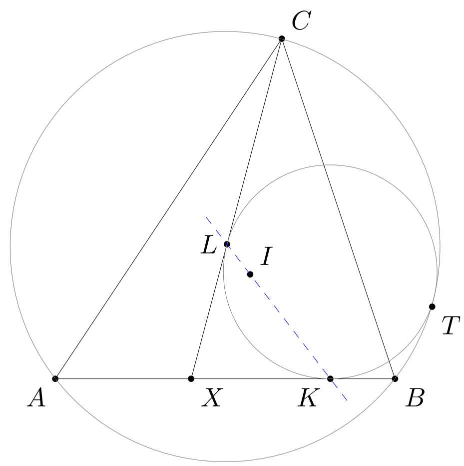
Proof. This again is no less trivial than the last statement, simply notice that $MD \cdot ME = MC^2 = MI^2$, thus (IDE) touches AM. Consequently,

$$\angle IDE = \angle AIE = \angle ALE \tag{3}$$

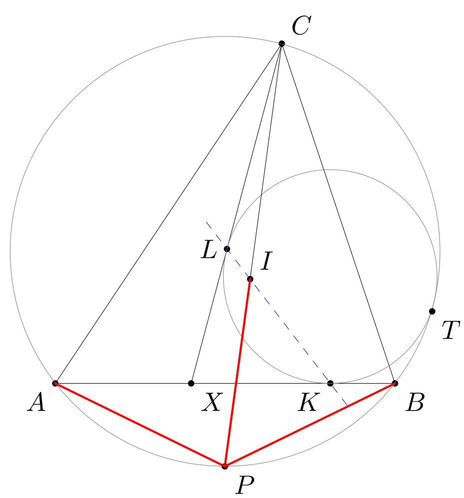
due to $AILE$ being cyclic. Which implies that AL touches ω. ∎

2 Sawayama's and Verrier's Lemma's

Theorem 3. ***(Sawayama's lemma)*** *Let $\triangle ABC$ be inscribed into Ω and let X be an arbitrary point AB. Consider ω which is tangent to the segment XC, the segment XB and Ω. Let L and K be the tangency points of ω with XC and XB respectively. Prove that L, K and I (the incenter of $\triangle ABC$) are colinear.*



Proof. Now it is time to utilize all the lemmas proven in the previous section. The first step is to extend CI till the intersection with Ω, let that intersection point be P. Then, due to the Trillium theorem it is clear that $PI = AP = PB$. This allows one to apply the last lemma to this configuration. Here C is serves as the arbitrary point from the last lemma and due to $IP = AP = PB$ it must be that the intersection of IK with ω, let that point be L', must be the tangency point from C to ω.



However, that tangency point is L by definition, thus $L' = L$ and consequently L, I and K are colinear.

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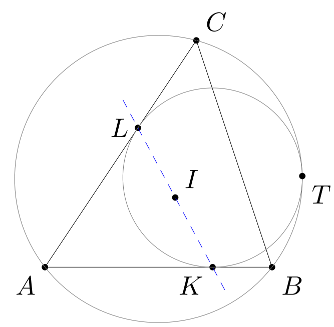
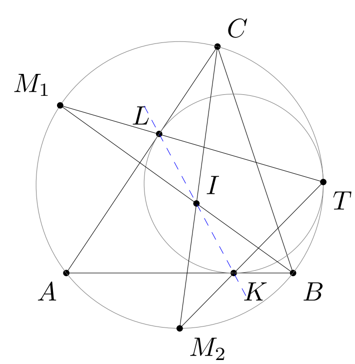
Now consider what happens when one moves X along AB, specifically let $X = A$. Then, Sawayama's lemma will transform itself into Verrier's lemma.

Theorem 4. ***(Verrier's lemma)*** *Let $\triangle ABC$ be inscribed into Ω and let ω be a circle which is tangent to the segments BC, AB and Ω at points L, K and T respectively. Then, L, K and I (incenter of $\triangle ABC$) are colinear.*



This statement has other proofs which do not involve Sawayama's lemma, one of the most notable ones is the following which showcases the mechanism at play.

Proof. Let us intersect TL and TK with Ω in points M_1 and M_2, by the shooting lemma it must be that M_1 and M_2 are the middle's of arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$. Consequently they must lie on the lines BI and CI.



All that is left is to apply Pascal's theorem for M_1AM_2BTC and conclude that L, I and K are colinear.

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A beautiful lemma about the M_1M_2 is the following,

Lemma 2. *The radical axis of $(A, 0)$ (the circle centered at A with a radius of zero) and ω is M_1M_2.*

Proof. Notice that due to the shooting lemma and its consequences, it must be that $M_1A^2 = M_1L \cdot M_1T$ and $M_2A^2 = M_2K \cdot M_2T$. This, means that,

$$pow_{(A,0)}M_1 = M_1^2 = M_1L \cdot M_1T = pow_\omega M_1 \tag{4}$$

$$pow_{(A,0)}M_2 = M_2^2 = M_2K \cdot M_2T = pow_\omega M_2 \tag{5}$$

Thus, it must be that M_1 and M_2 lie on the radical axis of $(A, 0)$ and ω, in other words M_1M_2 is the radical axis of $(A, 0)$ and ω.

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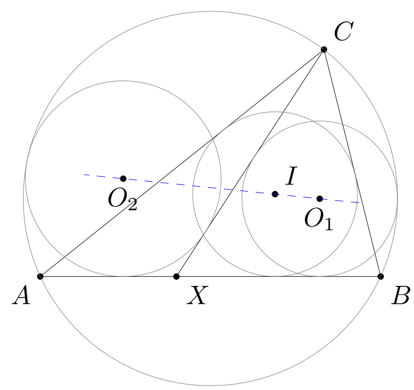
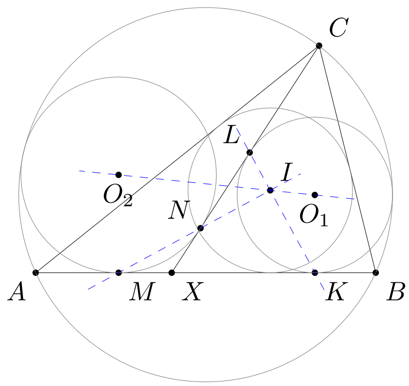
3 Thébault's theorem

Theorem 5. ***(Thébault's theorem)*** *If $\triangle ABC$ is inscribed into Ω, let ω_1 be the circle tangent to the segments XB and XC and Ω and let the circle ω_2 be the circle tangent to the segments XC, AX and Ω. Let O_1 and O_2 be the centers of ω_1 and ω_2. Then, O_1, O_2 and I (the incenter of $\triangle ABC$) are colinear.*



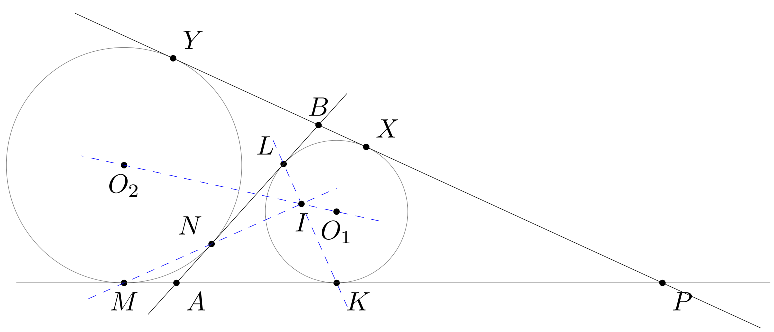
Proof. Notice, due to Sawayama's lemma it must be that $LK \cap MN = I$, where M, N are the tangency points of ω_2 with AX and XC and K, L are the tangency points of ω_1 with XB and XC. With this in mind I suggest looking at the problem from another perspective, LN is the inner tangent between ω_1 and ω_2 and MK is the outer tangent between ω_1 and ω_2. One must prove that $MN \cap LK \in O_1O_2$.



As it turns out this statement is true for arbitrary circles ω_1 and ω_2.

Lemma 3. *Let ω_1 and ω_2 be arbitrary circles prove that if LN is the inner tangent line between them and MK is the outer, then $MN \cap LK \in O_1O_2$.*

This is a wonderful statement to consider on its own. Proving this statement automatically proves Thébault's theorem. Let us consider the homothepy center P which transforms ω_1 to ω_2 (in other words the intersection of the two outer common tangents). Let X and Y be the points of tangency of the common tangent of ω_1 and ω_2. Let A and B be the intersection of NL with MK and XY respectively. Consider the triangle $\triangle ABP$, then ω_1 is its incircle and ω_2 is its excircle. Then, by the Iran lemma the projection of B onto the bisector of $\angle BPA$ must lie on both MN and LK.



In other words, the projection of B onto the bisector of $\angle APB$ is $MN \cap LK$. However, the projection of B onto the bisector of $\angle APB$ obviously is part of O_1O_2. Thus, $MN \cap LK \in O_1O_2$ and the lemma is proven, proving Thébault's theorem.

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