The Pressure Law in Two and Three Dimensions

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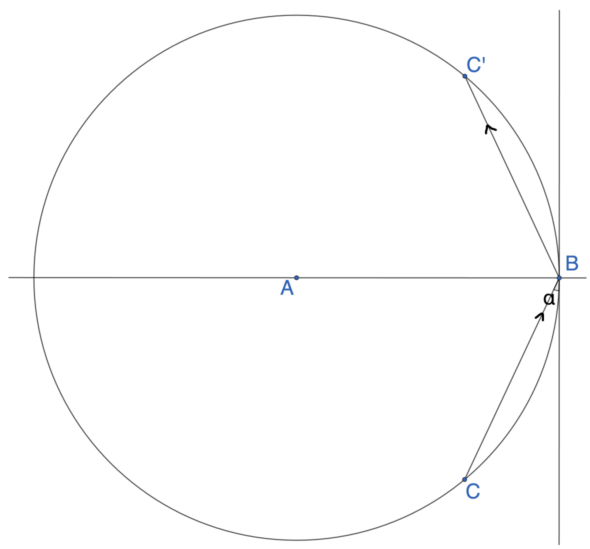
January 14, 2025

1 Two Dimensions, The Circle Case

Let us consider a two-dimensional particle moving around in a circle of radius R, and assume that all the collisions are perfectly elastic. Then, the trajectory of every particle inside the circle is completely characterized by an angle α, representing the angle between the line of trajectory and the tangent to the circle at the point of collision.



First, let us calculate the change in momentum. This can easily be done in the following way:

$$\Delta\vec{p} = m(\vec{v}_{end} - \vec{v}_{start}) = m \cdot 2V \cos\left(\frac{\pi}{2} - \alpha\right) = 2mV \sin\alpha \tag{1}$$

where V is the speed of the particle. Now, notice that due to the law of sines, it is simple to calculate the length of one segment, from one collision to the next (e.g., BC and BC' in the figure above):

$$L = \frac{R}{\sin(90-\alpha)} \cdot \sin(2\alpha) = \frac{R\sin(2\alpha)}{\cos\alpha} \tag{2}$$

This means that the time between collisions is always $\frac{L}{V}$, so it is simple to now find the applied pressure on the entire circle:

$$\begin{aligned} P = \frac{F}{D} = \frac{J}{D\cdot\Delta t} = \frac{2mV\sin\alpha}{D\cdot\frac{L}{V}} &= [2mV\sin\alpha] \,/\, \left(D\cdot\frac{R\sin(2\alpha)}{V\cos\alpha}\right) \\ = \frac{2mV^2\sin\alpha\cos\alpha}{D\cdot R\sin(2\alpha)} &= \frac{2mV^2\sin\alpha\cos\alpha}{2\pi R^2\sin(2\alpha)} = \frac{mV^2\sin\alpha\cos\alpha}{A\sin(2\alpha)} \\ &= \frac{mV^2\sin\alpha\cos\alpha}{2A\sin\alpha\cos\alpha} = \frac{1}{2}\frac{mV^2}{A} \end{aligned} \tag{3}$$

where A is the area of the circle (it simplifies out of $2\pi R^2$). Now, let us consider $N \to \infty$ particles, and then we obtain the following:

$$P = \frac{1}{2}\frac{MV_{av}^2}{A} = \frac{1}{2}\rho V_{av}^2 \tag{4}$$

where V_{av}^2 is the average square speed of the particles, M is the total mass of N particles, and ρ is the density of the gas.

2 Three Dimensions, The Sphere Case

Let us now consider a particle in three dimensions, confined in a sphere. Its trajectory is a circle, which is a subset of the sphere's surface. In fact, this circular subset must be centered at the center of the sphere. Let us assume that the particles are uniformly distributed in the sphere, then:

$$P = \frac{F}{A} = \frac{J}{A\cdot\Delta t} = \frac{2mV^2}{2A\cdot R} = \frac{2mV^2}{8\pi R^3} = \frac{1}{3}\cdot\frac{mV_{av}^2}{V_R} \tag{5}$$

where V_R is the volume of the sphere of radius R. Again, taking this to $N \to \infty$ particles, and noticing that $\frac{m}{V_R}$ is simply the density, we obtain the well-known pressure law:

$$P = \frac{1}{3}\rho V_{av}^2 \tag{6}$$

3 N-Dimensions

Let us now try calculating for n-dimensions:

$$P = \frac{mV_{av}^2}{D\cdot R} \tag{7}$$

where D is the hypersurface area of an n-dimensional sphere, S_3. This formula, with the exact same reasoning as for three dimensions, comes from the fact that the trajectory will be enclosed in a circle. Taking this to n dimensions, we quickly see the following formula:

$$P_n = \frac{mV_{av}^2}{S_{n-1}R} = \frac{mV_{av}^2}{R} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{2R^{n-1}\pi^{n/2}} = \frac{mV_{av}^2}{2R^n} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}} \tag{8}$$

To make the formula a bit more elegant, let us substitute the value for the volume of the n-dimensional sphere, and then we get:

$$\begin{aligned} P = \frac{mV_{av}^2}{2R^n} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}} = \left\{ V_{R,n} = \frac{R^n \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \right\} \\ = \frac{mV_{av}^2}{2V_{R,n}} \cdot \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}} = \frac{mV_{av}^2}{2V_{R,n}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \\ = \frac{mV_{av}^2}{2V_{R,n}} \cdot \frac{2}{n} = \frac{1}{n} \cdot \frac{mV_{av}^2}{V_{R,n}} = \frac{1}{n}\rho V_{av}^2 \end{aligned} \tag{9}$$

Thus, at least for spheres, the general formula is $P = \frac{1}{n}\rho V_{av}^2$.