Geometry Marathon P220

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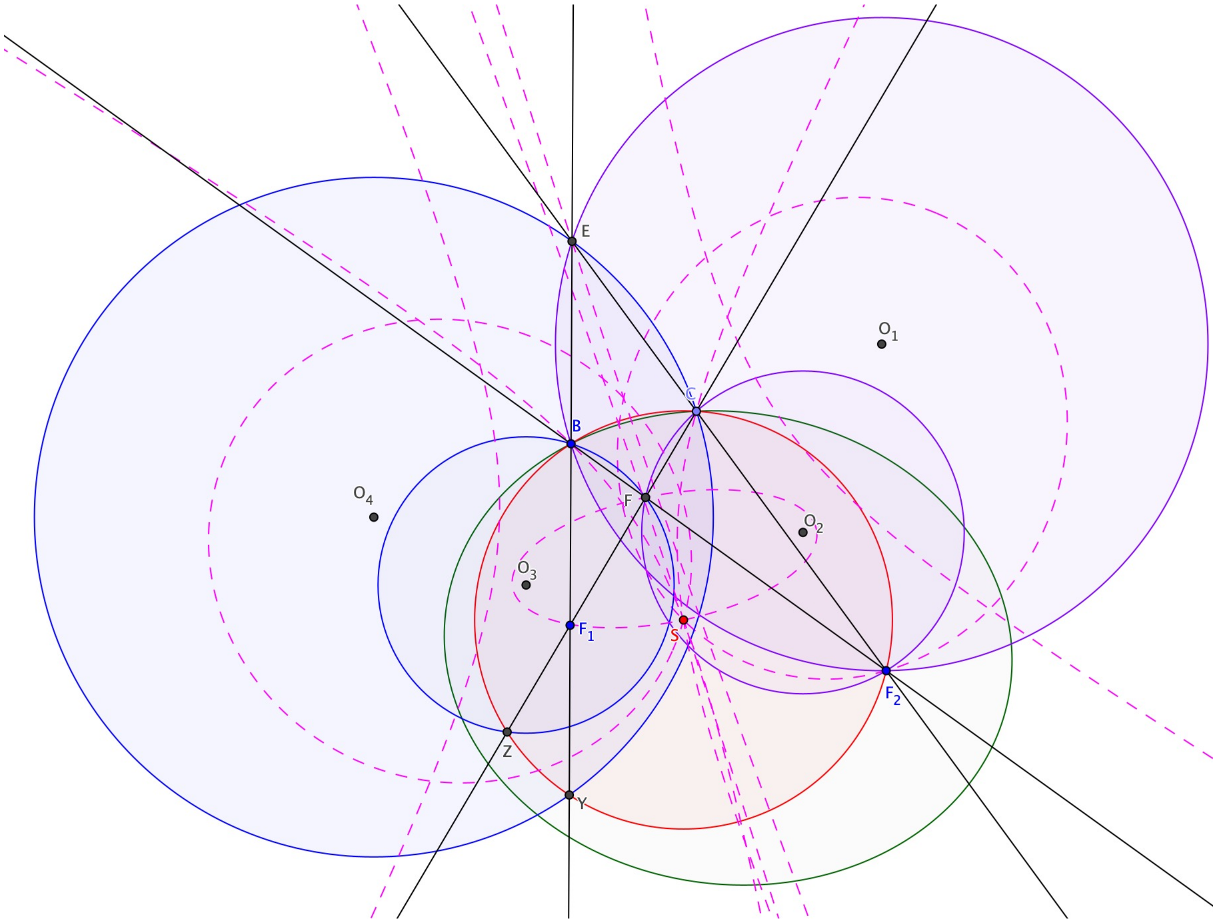
1 Case Reduction

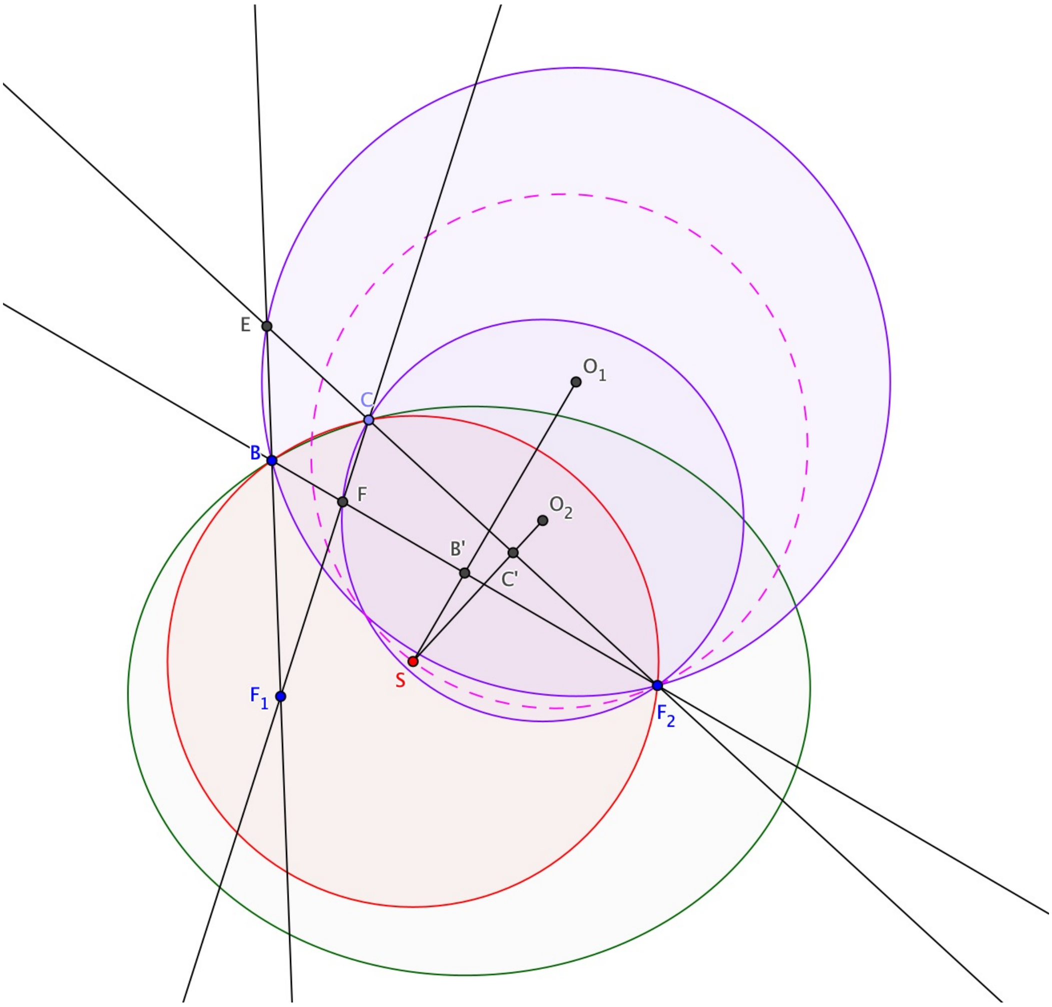


Notice that the centers of circles tangent to two given ones lie on a conic whose foci are the centers of the respective circles. Thus, the problem is equivalent to finding a circle ω with center O such that for every pair $\binom{4}{2} = 6$ of circles O lies on the generated conic.

Throughout this proof I will be showing that the desired point O is actually the center of the circle (BCF_2), let that center be S. What will follow is a careful analysis of the different pairs of circles and a proof why S lies on the respective conic.

Notice that we don't actually need all pairs of circles, consequently let us not prove the statement for (O_4, O_3) (it will follow) and reduce all the other cases to two cases (O_1, O_2) and (O_2, O_4). Every other pair is done analogously to the case (O_2, O_4).

2 O_1O_2 case



Let $B' = BF_2 \cap SO_1$ and $C' = SO_2 \cap CF_2$, then to show that S lies on the ellipse with foci O_1 and O_2 we want to show that,

$$SO_1 + SO_2 \stackrel{?}{=} R_1 + R_2$$

however notice that,

$$SO_1 + SO_2 = SB' + B'O_1 + SC' + C'O_2$$

however,

$$\begin{cases} SB' = \tan(90 - \angle BSF_2/2) \cdot \frac{BF_2}{2} = \cot(\angle ECB) \cdot \frac{BF_2}{2} \\ O_1B' = \tan(90 - \angle BO_1F_2/2) = \cot(\angle BEF_2) \cdot \frac{BF_2}{2} \\ SC' = \tan(90 - \angle CSF_2/2) = \cot(\angle CBF_2) \cdot \frac{CF_2}{2} \\ C'O_2 = \tan(90 - \angle CO_2F_2/2) = \cot(\angle BFC) \cdot \frac{CF_2}{2} \end{cases}$$

thus we want to show that,

$$R_1 + R_2 = \frac{CF_2}{2}\Big(\cot(\angle CBF_2) + \cot(\angle BFC)\Big) + \frac{BF_2}{2}\Big(\cot(\angle ECB) + \cot(\angle BEF_2)\Big)$$

now using the Law of Sines we know that,

$$R_1 + R_2 = \frac{1}{2\sin(\angle CF_2B)}(CF + EB)$$

now let us use that $\cot(A) + \cot(B) = \frac{\sin(180-A-B)}{\sin(A)\sin(B)}$ and $\cot(A) - \cot(B) = \frac{\sin(B-A)}{\sin(B)\sin(A)}$, then,

$$\frac{CF_2}{2}\Big(\cot(\angle CBF_2) + \cot(\angle BFC)\Big) = \frac{CF_2}{2} \cdot \frac{\sin(\angle BCF_1)}{\sin(\angle CBF_2)\sin(\angle BFC)} = \frac{CF_2}{2\sin(\angle CBF_2)} \cdot \frac{BF}{BC}$$

however by the Law of Sines we know that,

$$\frac{CF_2}{\sin(\angle CBF_2)} = \frac{BC}{\sin(\angle EF_2B)}$$

thus,

$$\frac{CF_2}{2\sin(\angle CBF_2)} \cdot \frac{BF}{BC} = \frac{BC}{2\sin(\angle EF_2B)} \cdot \frac{BF}{BC} = \frac{BF}{2\sin(\angle EF_2B)}$$

Similarly,

$$\frac{BF_2}{2}\Big(\cot(\angle ECB) + \cot(\angle BEF_2)\Big) = \frac{BF_2}{2} \cdot \frac{\sin(\angle EBC)}{\sin(\angle ECB)\sin(\angle BEF_2)} = \frac{BF_2}{2\sin(\angle BEF_2)} \cdot \frac{EC}{EB}$$

however by the Law of Sines we know that,

$$\frac{BF_2}{\sin(\angle BEF_2)} = \frac{EB}{\sin(\angle EF_2B)}$$

thus,

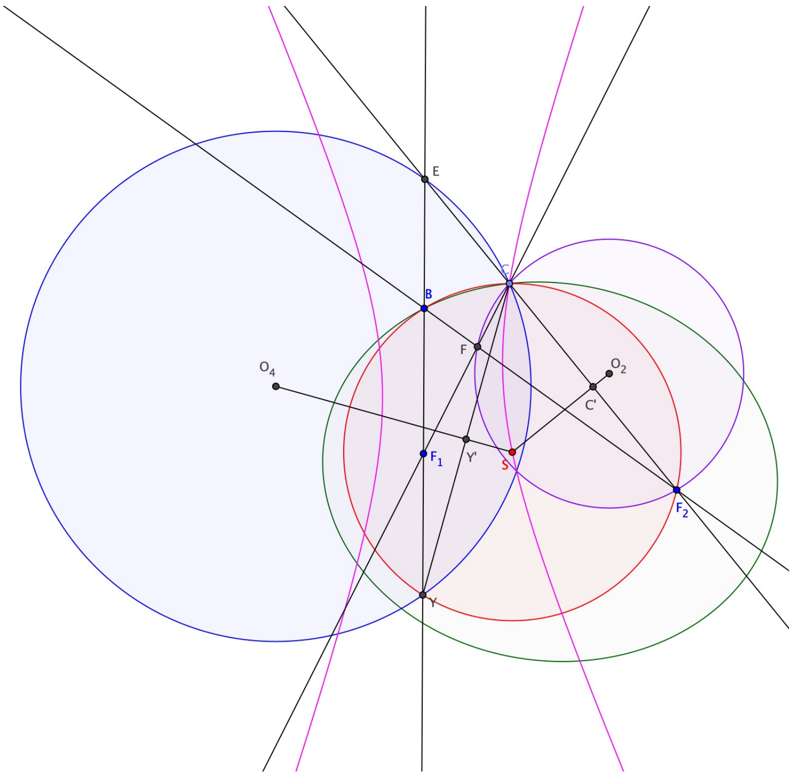
$$\frac{BF_2}{2\sin(\angle BEF_2)} \cdot \frac{EC}{EB} = \frac{EB}{2\sin(\angle EF_2B)} \cdot \frac{EC}{EB} = \frac{EC}{2\sin(\angle EF_2B)}$$

Consequently the problem is now reformulated as,

$$\frac{1}{2\sin(\angle EF_2B)}\Big(BF + EC\Big) \stackrel{?}{=} \frac{1}{2\sin(\angle CF_2B)}(CF + EB)$$

which is the same as $BF + EC = CF + EB$ which is equivalent to $EBCF$ being circumscribed which is well-known.

3 O_4O_2 case



Notice that to show that S lies on the desired hyperbola we need to show that,

$$SO_4 + SO_2 \stackrel{?}{=} R_2 + R_4$$

now let us express SO_4 and SO_2 (by introducing $Y = SO_4 \cap CY$ and $C' = CF_2 \cap SO_2$)

$$\begin{cases} SC' = \tan(90 - \angle CSF_2/2) \cdot \frac{CF_2}{2} = \cot(\angle F_2BC) \cdot \frac{CF_2}{2} \\ C'O_2 = \tan(90 - \angle CO_2F_2/2) \cdot \frac{CF_2}{2} = \cot(180 - \angle CFF_2) \cdot \frac{CF_2}{2} = \cot(\angle BFC) \cdot \frac{CF_2}{2} \\ SY' = \tan(180 - \angle CSY/2) \cdot \frac{CY}{2} = \cot(\angle CF_2Y) \cdot \frac{CY}{2} = \cot(\angle EBC) \cdot \frac{CY}{2} \\ O_4Y' = \tan(180 - \angle YO_4C/2) \cdot \frac{CY}{2} = \cot(\angle YEC) \cdot \frac{CY}{2} \end{cases}$$

similarly by the Law of Sines we know that,

$$R_2 - R_4 = \frac{CF}{2\sin(\angle CF_2B)} - \frac{EC}{2\sin(\angle EYC)} = \frac{1}{2\sin(\angle BF_2C)}(CF - EC)$$

now using the Law of Sines again let us notice that,

$$CY = R\sin\angle CF_2Y = \frac{CF_2\sin(CF_2Y)}{\sin(\angle CBF_2)} = \frac{CF_2\sin(\angle EBC)}{\sin(\angle CBF_2)}$$

this allows us to rewrite the previous expression to obtain the reformulated problem,

$$\frac{CF_2}{2}\left(\cot(\angle F_2BC)+\cot(\angle BFC)\right) - \frac{CF_2}{2}\cdot\frac{\sin(\angle EBC)}{\sin(\angle CBF_2)}\left(\cot(\angle EBC)+\cot(\angle YEC)\right) = \frac{1}{\sin(BF_2C)}(CF-EC)\ (\star)$$

now let us use the fact that $\cot(A)+\cot(B) = \frac{\sin(180-A-B)}{\sin(A)\sin(B)}$, then,

$$\frac{CF_2}{2}\left(\cot(\angle F_2BC)+\cot(\angle BFC)\right) = \frac{CF_2}{2}\cdot\frac{\sin(\angle BCF)}{\sin(\angle F_2BC)\sin(\angle BFC)} = \frac{CF_2}{2\sin(\angle F_2BC)}\cdot\frac{BF}{BC}$$

similarly,

$$\frac{CF_2}{2}\cdot\frac{\sin(\angle EBC)}{\sin(\angle CBF_2)}\left(\cot(\angle EBC)+\cot(\angle YEC)\right) = \frac{CF_2}{2}\frac{\sin(\angle EBC)}{\sin(\angle CBF_2)}\cdot\frac{\sin(\angle ECB)}{\sin(\angle YEC)\sin(\angle EBC)} = \frac{CF_2}{2\sin(\angle F_2BC)}\cdot\frac{EB}{BC}$$

thus we can rewrite $(\star)$ as,

$$\frac{CF_2}{2}\cdot\frac{BF-EB}{BC} = \frac{\sin(\angle F_2BC)}{2\sin(\angle BF_2C)}(CF-EC) = \frac{CF_2}{2BC}(CF-EC)$$

but this is the same as $BF-EB = CF-EC$, which is equivalent to $EBFC$ being circumscribed, however this is a known lemma.