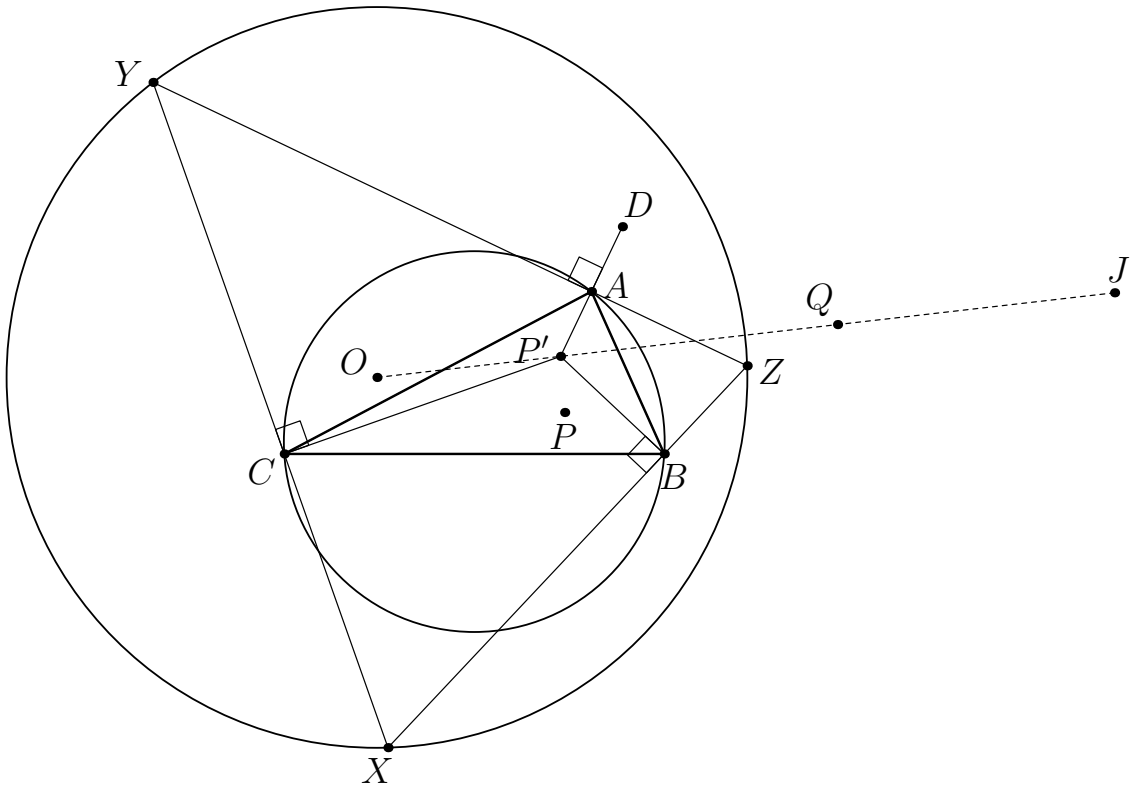
Geometry Marathon P231

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Problem 1 (Geometry Marathon P231)

Let P be a finite point not on the sidelines of $\triangle ABC$, and let P' be its isogonal conjugate. Let XYZ be the antipedal triangle of P'. If Q and J are the inverses of P in (ABC) and of P' in (XYZ), respectively, prove that J, Q, P' are collinear and that one of them is the midpoint of the other two.



Proof. Let M be defined as the midpoint of $P'J$, one may reformulate the problem as showing that M is the inverse of P in (ABC), which is equivalent to (ABC) being the Apollonius circle of P and M and consequently it will suffice to show that $\frac{PA}{MA} = \frac{BP}{BM} = \frac{CP}{CM}$.

Lemma 1 $\triangle PAB \sim \triangle P'YX$

Proof. Since $\angle PAB = \angle CAP' = \angle XYP'$ and $\angle PBA = \angle CBP' = \angle YXP'$ one obtained the desired similarity! ■

This implies that,

$$PA = \frac{AB}{XY} \cdot P'Y = \frac{P'Z \sin \angle YZX \cdot P'Y}{2R \sin \angle YZX} = \frac{P'Z \cdot P'Y}{2R}$$

where R is the circumradius of $\triangle XYZ$! Now in order to calculate MA, let us find DJ and apply homothety centered at P' with coefficient 2 (where D is the reflection of P' over the line YZ).

Lemma 2 $\triangle JYD \sim \triangle ZOP'$

Proof. Indeed since $\triangle JYO \sim \triangle YP'O$, thus $\frac{JY}{DY} = \frac{JY}{P'Y} = \frac{R}{OP'}$ (since (XYZ) is the Apollonius circle wrt P' and J) and,

$$\begin{aligned}\angle JYD &= \angle JYO - \angle OYP' - 2\angle P'YA = \angle OP'Y - \angle OYP' - 2\angle YP'A \\ &= \angle(OP', YZ) - \angle OYZ = \angle(OP', YZ) - \angle OZY = \angle ZOP'\end{aligned}$$

thus by the angle and the ratio one obtains the desired similarity! ■

Thus,

$$DJ = P'Z \cdot \frac{YD}{OP'} = \frac{P'Z \cdot P'Y}{OP'}$$

thus $\frac{PA}{MA} = \frac{OP'}{R}$ which is symmetric and thus the same as $\frac{BP}{BM}$ and $\frac{CP}{CM}$! ■