Ramsey's Numbers, R(5, 5)

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1 Analysing the Components in the Graph

This document aims to find a better estimate for the values of $R(5,5)$. First, let G be a graph with the maximum number of nodes so there is no K_5 and no K_5 in the complement of G. Then, it is trivial to see that

$$comp(G) < 5 \tag{1}$$

It is known that $42 \leq |G| \leq 47$. This means if $comp(G) = 4$, we would have a component with the size $\left\lceil \frac{n}{4} \right\rceil \geq 11$, and because there is no way of picking 5 nodes which are not connected, we conclude all 4 components must be complete, meaning G contains K_{11} which contains K_5, contradiction. This argument allows us to improve the initial statement.

$$comp(G) < 4 \tag{2}$$

Two of the three components must be complete if $comp(G) = 3$. In such cases, there should be a component with $\left\lceil \frac{n}{3} \right\rceil \geq 14$ nodes. If this component is one of the complete ones, we have found K_5, so let us assume this component is the one that is not complete. Then, the other two components can't be greater than 4 because else we would have found a K_5, meaning that the size of the selected component C is at least $42 - 4 \cdot 2 = 34$ nodes. Zarankiewicz's lemma tells us there exists a node with a degree at most $\left\lfloor \frac{k-2}{k-1} \cdot n \right\rfloor \leq 25$, indicating that there exists a node which isn't connected to the other 8 nodes in the same component. However, if any of the 8 nodes were not connected, we could select 2 from the complete components, the Zarankiewicz node, and the two nodes not connected from the 8 nodes. We would have found a K_5 in the complement of G, meaning the 8 nodes create a complete graph K_8, meaning they contain a K_5, contradiction. Now, we bring the upper bound of the number of components down to two.

$$comp(G) < 3 \tag{3}$$

We can break this limit even further by noticing if we have two components. There are two options: one is complete, the other is not, or both are incomplete. If only one is complete, the other component will have at least 38 nodes, even more than the situation discussed for the 3 components. This means that the only option worth looking at is if both still need to be completed, but then at least one of them contains at least 21 nodes. There is a Zarankiewicz's node with at most $\left\lfloor \frac{k-2}{k-1} \cdot n \right\rfloor \leq 15$ connections, meaning it is not connected with another 5 nodes applying the same argument as in the previous situation we conclude or there is a K_5 in one of the components or there are three nodes in one component which are not connected adding another two from the other not complete component we again find a K_5 in the complement of G.

This means G is connected.